So far, we have come across several classes of continuous functions:

i) "constant functions". Let $c \in \mathbb{R}$, then
$$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x) = c$$

ii) "identity map".
$$id_{\mathbb{R}}: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto x$$

iii) "polynomials". Let $a_0, a_1, \dots, a_n \in \mathbb{R}$, then
$$p: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto p(x) := a_n x^n + \dots + a_1 x + a_0$$

iv) "rational functions". Let
$$P(x) = a_n x^n + \dots + a_1 x + a_0, \quad Q(x) = b_m x^m + \dots + b_1 x + b_0$$
Then
$$R: D \longrightarrow \mathbb{R}, \quad x \mapsto R(x) = \frac{P(x)}{Q(x)} \text{ is continuous}$$
on $D := \{x \in \mathbb{R} \mid Q(x) \neq 0\}$.

Now let's see how to define $\sqrt[n]{x}$, $\mathrm{Exp}(x)$ and $\mathrm{Log}(x)$.

<u>Corollary 4.2:</u>

Let $I \subset \mathbb{R}$ be an interval (include cases $[a, +\infty)$, $(-\infty, b]$, $(-\infty, +\infty)$) and let $f: I \to \mathbb{R}$ be a continuous function. Then $f(I) \subset \mathbb{R}$ is also an interval.

<u>Proof:</u>

We set $B := \sup f(I) \in \mathbb{R} \cup \{+\infty\}$, $A := \inf f(I) \in \mathbb{R} \cup \{-\infty\}$ and show that $(A, B) \subset f(I)$. Let $y \in \mathbb{R}$ be an

arbitrary number with $A < y < B$.
According to definition of A and B, there exist numbers $a, b \in I$ s.t.

$$f(a) < y < f(b).$$

Corollary 4.1 $\Rightarrow$ $\exists\, x \in I$ with $f(x) = y$, therefore $y \in f(I)$. Thus $(A, B) \subset f(I)$.
$\Rightarrow$ $f(I)$ is of the form:

$$(A, B),\ (A, B],\ [A, B),\ \text{or } [A, B]$$

$\square$

<u>Definition 4.7:</u>

$f: [a, b] \to \mathbb{R}$ is called "strictly monotonically increasing", if the following holds:

$$a \leq x < y \leq b \Rightarrow f(x) < f(y)$$

<u>Proposition 4.6:</u>

Let $D \subset \mathbb{R}$ be an interval and $f: D \to \mathbb{R}$ be a continuous, and strictly monotonically increasing (or decreasing) function. Then, for $D' := f(D)$, $f: D \to D'$ is bijective, and the inverse function $f^{-1}: D' \to \mathbb{R}$ is also continuous and strictly monotonically increasing (or decreasing).

Proof:

Corollary 4.2 shows that $D' = f(D)$ is again an interval. f is injective by Definition 4.7 and surjective by Prop. 4.5.

$\Rightarrow$ f is bijective and f^{-1} is strictly monotonically increasing/decreasing

Need to show continuity:

Let $b \in D'$ be given and $a = f^{-1}(b)$.

Suppose b is not a boundary point of D'

$\Rightarrow$ a is not a boundary point of D.

Without loss of generality: $[a-\varepsilon, a+\varepsilon] \subset D$

Set $b_1 := f(a-\varepsilon)$ and $b_2 := f(a+\varepsilon)$

$\Rightarrow$ $b_1 < b < b_2$, and $f : [a-\varepsilon, a+\varepsilon] \longrightarrow [b_1, b_2]$ bijective

Let $\delta := \min(b-b_1, b_2-b)$. Then

$$f^{-1}((b-\delta, b+\delta)) \subset (a-\varepsilon, a+\varepsilon)$$

$\Rightarrow$ f^{-1} is continuous in b (ε-δ criterion).

Proceed analogously for $b \in D'$ boundary point ($a = f^{-1}(b)$ is then boundary point of D) $\square$

Example 4.8:

Let $f: [0,1) \cup [2,3] \longrightarrow \mathbb{R}$ be given as follows

$$f(x) = \begin{cases} x, & 0 \leq x < 1, \\ x-1, & 2 \leq x \leq 3 \end{cases}$$

f is continuous and monotonically increasing, but $f^{-1}: [0,2] \longrightarrow \mathbb{R}$ is discontinuous at $y=1$. So the requirement that the domain of definition is an interval is essential!

Example 4.9 :

i) Let $n \in \mathbb{N}$. The power function $\mathbb{R} \ni x \longmapsto x^n \in \mathbb{R}$ is continuous according to Prop. 4.4, and is monotonically increasing on $\mathbb{R}_+ = (0, \infty)$. Prop. 4.6 then implies that the nth "root function"

$$\mathbb{R}_+ \ni y \longmapsto \sqrt[n]{y} \in \mathbb{R}_+$$

is continuous.

ii) Consider the function $\mathrm{Exp}: \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\mathrm{Exp}(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

This series converges due to the quotient criterion of Prop. 3.10, namely we have for $a_k := \frac{x^k}{k!} \neq 0$:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|x|}{k+1} \longrightarrow 0 \quad (k \to \infty)$$

This convergent series defines the "Exponential function".

It has the following property:

$$\mathrm{Exp}(x)\,\mathrm{Exp}(y) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!}$$

$$= \sum_{k,\ell=0}^{\infty} \frac{x^k y^\ell}{k!\,\ell!} = \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{\infty} \frac{x^k y^\ell}{k!\,\ell!}\right) = (*)$$

Substitute for fixed k the index ℓ by the summation variable $n := k+\ell$; that is substitute ℓ by $n-k$. We obtain

$$(*) = \sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} \frac{x^k y^{n-k}}{k!\,(n-k)!}\right) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \binom{n}{k} \frac{x^k y^{n-k}}{n!}$$

Exchange of summation order finally gives

$$(*) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right)}_{=(x+y)^n} = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

$$\Longrightarrow \mathrm{Exp}(x) \cdot \mathrm{Exp}(y) = \mathrm{Exp}(x+y)$$

This is called the "addition theorem".

<u>Claim:</u>

Exp > 0, Exp is continuous and monotonically increasing with $\mathrm{Exp}(\mathbb{R}) = (0, \infty)$.

<u>Proof:</u>

The addition theorem gives

$$\forall x \in \mathbb{R} : \mathrm{Exp}(x) = \left(\mathrm{Exp}\left(\tfrac{x}{2}\right)\right)^2 \geq 0,$$

and due to

$$\mathrm{Exp}(x)\,\mathrm{Exp}(-x) = 1 \quad (\Rightarrow \mathrm{Exp}(x) \neq 0)$$

we have $\mathrm{Exp}(x) > 0 \;\; \forall\, x \in \mathbb{R}$.

Further, we have for $|h| < 1$

$$|\mathrm{Exp}(h) - 1| = \left| \sum_{k=1}^{\infty} \frac{h^k}{k!} \right| \leq \sum_{k=1}^{\infty} |h|^k$$

$$= \frac{|h|}{1-|h|} \longrightarrow 0 \quad (h \to 0),$$

so for $x = x_0 + h \longrightarrow x_0$ we get

$$\mathrm{Exp}(x) - \mathrm{Exp}(x_0)$$
$$= \mathrm{Exp}(x_0)\big(\mathrm{Exp}(h) - 1\big) \longrightarrow 0, \quad (**)$$

and the function Exp is continuous.

As
$$\mathrm{Exp}(h) - 1 = \sum_{k=1}^{\infty} \frac{h^k}{k!} > 0 \quad \text{for } h > 0,$$

$(**)$ gives the desired monotomy:

$$\mathrm{Exp}(x_0) < \mathrm{Exp}(x) \quad \text{for } x_0 < x = x_0 + h.$$

Finally, we apparently have
$$\mathrm{Exp}(x) \to \infty \quad (x \to \infty);$$
together with $\mathrm{Exp}(-x) = \dfrac{1}{\mathrm{Exp}(x)}$ we get

$$\mathrm{Exp}(x) = \frac{1}{\mathrm{Exp}(-x)} \longrightarrow 0 \quad (x \to -\infty)$$

According to Prop. 4.6 the function $\mathrm{Exp} : \mathbb{R} \to (0, \infty)$ then has a continuous inverse function

$$\mathrm{Log}\,(\mathrm{Exp}|_{\mathbb{R}})^{-1} : (0, \infty) \longrightarrow \mathbb{R}$$

Due to

$$\mathrm{Exp}\big(\mathrm{Log}(x)+\mathrm{Log}(y)\big) = \mathrm{Exp}\big(\mathrm{Log}(x)\big)\cdot \mathrm{Exp}\big(\mathrm{Log}(y)\big)$$
$$= x\,y$$

we get the "Addition theorem":

$$\forall\, x, y > 0: \ \mathrm{Log}(xy) = \mathrm{Log}(x) + \mathrm{Log}(y)$$